

Glueball operators and the microscopic approach to $\mathcal{N} = 1$ gauge theories

Frank Ferrari, Stanislav Kuperstein and Vincent Wens

*Service de Physique Théorique et Mathématique,
Université Libre de Bruxelles and International Solvay Institutes,
Campus de la Plaine, CP 231, B-1050 Bruxelles, Belgique
E-mail: frank.ferrari@ulb.ac.be, skuperst@ulb.ac.be, vwens@ulb.ac.be*

ABSTRACT: We explain how to generalize Nekrasov's microscopic approach to $\mathcal{N} = 2$ gauge theories to the $\mathcal{N} = 1$ case, focusing on the typical example of the $U(N)$ theory with one adjoint chiral multiplet X and an arbitrary polynomial tree-level superpotential $\text{Tr } W(X)$. We provide a detailed analysis of the generalized glueball operators and a non-perturbative discussion of the Dijkgraaf-Vafa matrix model and of the generalized Konishi anomaly equations. We compute in particular the non-trivial quantum corrections to the Virasoro operators and algebra that generate these equations. We have performed explicit calculations up to two instantons, that involve the next-to-leading order corrections in Nekrasov's Ω -background.

KEYWORDS: Duality in Gauge Field Theories, Nonperturbative Effects, Supersymmetric gauge theory, Gauge-gravity correspondence.

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1. Introduction

Recently, a very general strategy to derive non-perturbative exact results in $\mathcal{N} = 1$ gauge theories from a microscopic point of view was explained [1]. The starting point is to consider the gauge theory path integral with arbitrary boundary conditions at infinity. A microscopic quantum effective superpotential W_{mic} can be derived as a function of the boundary conditions. This effective superpotential has two fundamental properties. First,

it can always be computed exactly in a semi-classical instanton framework by choosing the boundary conditions appropriately and then performing suitable analytic continuations. Second, the stationary points of W_{mic} describe *all* the quantum vacua of the theory, including the strongly coupled confining vacua. A direct procedure for solving the theory in the chiral sector from microscopic instanton calculations then follows. In particular, the full power of Nekrasov's technology [2], which itself was the crowning achievement of many years of developments in instanton calculus [3–6] and which was successful in solving $\mathcal{N} = 2$ gauge theories [7], can be applied to the realm of $\mathcal{N} = 1$ gauge theories, generalizing useful early work [8].

The basic example on which to apply these ideas is the $\mathcal{N} = 1$ theory with gauge group $U(N)$, one adjoint chiral superfield X and an arbitrary polynomial tree-level superpotential $\text{Tr } W(X)$ such that

$$W'(z) = \sum_{k=0}^d g_k z^k = g_d \prod_{i=1}^d (z - w_i). \tag{1.1}$$

The solution of this model can be generalized to many other $\mathcal{N} = 1$ gauge theories with various gauge groups and matter contents. The usual approach is to use the Dijkgraaf-Vafa matrix model [9], or equivalently the generalized Konishi anomaly equations supplemented with an appropriate glueball effective superpotential [10]. These approaches have been motivated by some perturbative calculations [11, 10]. Here perturbative is with respect to the gauge coupling constant. Equivalently, the gauge field in [11, 10] is treated as an external classical background field. This is clearly inadequate to derive exact non-perturbative results. Our main interest is actually in computing the expectation values of various chiral operators, which do not have perturbative corrections!

In the present paper, we provide a non-perturbative check of the matrix model and the anomaly equations up to the second order in the instanton expansion. An exact proof to all orders, that applies to all the vacua of the theory, will be presented in a forthcoming paper [12]. Our explicit calculations show how remarkable it is for the anomaly equations to retain their perturbative form, at the expense of a non-perturbative redefinition of the variables as explained in [13]. In particular, the generators of the equations, which form perturbatively a truncated super-Virasoro algebra, get extremely strong quantum corrections due to the non-linearity of the associated transformations. Their action does not close in the chiral ring, and to obtain a closed algebra one needs to enlarge considerably the set of generators.

The full set of non-trivial expectation values in the theory (1.1) is given by [10]

$$u_n = \langle \text{Tr } X^n \rangle, \quad v_n = -\frac{1}{16\pi^2} \langle \text{Tr } W^\alpha W_\alpha X^n \rangle, \tag{1.2}$$

where W^α is the vector chiral superfield whose lowest component is the gluino field. It is convenient to work with the generating functions

$$R(z; \mathbf{a}, q) = \sum_{n \geq 0} \frac{u_n}{z^{n+1}}, \quad S(z; \mathbf{a}, \mathbf{g}, q) = \sum_{n \geq 0} \frac{v_n}{z^{n+1}}. \tag{1.3}$$

We have indicated explicitly the dependence on the couplings g_k , denoted collectively by \mathbf{g} , the instanton factor

$$q = \Lambda^{2N}, \tag{1.4}$$

and the boundary conditions at infinity for the chiral superfield X ,

$$X_\infty = \text{diag}(a_1, \dots, a_N) = \text{diag } \mathbf{a}. \tag{1.5}$$

The function $R(z; \mathbf{a}, q)$ does not depend on \mathbf{g} [8] and can be computed exactly using the results of [2, 7]. It was shown in [1] that, on the extrema of $W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q)$, $R(z)$ coincides with the result obtained from the matrix model. On the other hand, very little is known about the generalized glueball operators v_n for arbitrary \mathbf{a} and n (the case $n = 0$ was discussed in [1]). The study of the generating function $S(z; \mathbf{a}, \mathbf{g}, q)$ will thus be a central topic in the present work. An important goal is to show that it coincides with the matrix model prediction on-shell (i.e. on the extrema of W_{mic}).

The plan of the paper is as follows. In section 2, we explain the general set-up and introduce Nekrasov's Ω -background, the localization formulas and the sum over colored partitions that we use to perform our calculations. We have been very careful in obtaining the relevant equations, which can be found in the literature in many different, and often erroneous, forms. We give general formulas for the generating functions $R(z; \mathbf{a}, q)$, $S(z; \mathbf{a}, \mathbf{g}, q)$ and the microscopic quantum superpotential $W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q)$. In section 3, we present our explicit two-instanton calculations in the Ω -background. In section 4, we focus on the anomaly equations. After a general discussion of the non-perturbative properties of these equations, we derive the quantum generators and algebra that generate the equations. We show that the results are consistent with the Dijkgraaf-Vafa matrix model and glueball superpotential. We present our conclusions in section 5. A technical appendix is also included at the end of the paper.

2. General set-up

2.1 Quantum superpotential and correlators

The microscopic quantum superpotential $W_{\text{mic}}(\mathbf{a})$ is defined [1] by the following euclidean path integral with given boundary conditions at infinity (1.5),

$$e^{-\int d^4x (2N \text{Re} \int d^2\theta W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) + D\text{-terms})} = \int_{X_\infty = \text{diag } \mathbf{a}} d\mu e^{-\mathcal{S}_E}, \tag{2.1}$$

where \mathcal{S}_E is the euclidean super Yang-Mills action and $d\mu$ the path integral measure including the ghosts. It is shown in [1] that

$$W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) = \langle \mathbf{a} | \text{Tr } W(X) | \mathbf{a} \rangle, \tag{2.2}$$

where the expectation value $\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle$ of any chiral operator \mathcal{O} is defined by

$$\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle = \frac{\int_{X_\infty = \text{diag } \mathbf{a}} d\mu \mathcal{O} e^{-\mathcal{S}_E}}{\int_{X_\infty = \text{diag } \mathbf{a}} d\mu e^{-\mathcal{S}_E}} = \mathcal{O}(\mathbf{a}, \mathbf{g}, q). \tag{2.3}$$

Equation (2.2) follows from the $U(1)_R$ symmetry of the theory, for which the charges of the superspace coordinates θ^α , instanton factor q , chiral superfield X , vector superfield W^α , boundary conditions \mathbf{a} , couplings \mathbf{g} and superpotential W_{mic} are given by

$$\begin{array}{c}
 \theta^\alpha \quad q \quad X \quad W^\alpha \quad \mathbf{a} \quad \mathbf{g} \quad W_{\text{mic}} \\
 U(1)_R \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 2 \quad 2.
 \end{array} \tag{2.4}$$

By varying the highest components of the chiral superfields \mathbf{g} and q in (2.1), we derive the fundamental formulas

$$n \frac{\partial W_{\text{mic}}}{\partial g_{n-1}} = \langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle = u_n(\mathbf{a}, \mathbf{g}, q), \tag{2.5}$$

$$Nq \frac{\partial W_{\text{mic}}}{\partial q} = -\frac{1}{16\pi^2} \langle \mathbf{a} | \text{Tr } W^\alpha W_\alpha | \mathbf{a} \rangle = v_0(\mathbf{a}, \mathbf{g}, q). \tag{2.6}$$

The gauge theory expectation values are obtained by going on-shell,

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = 0. \tag{2.7}$$

These equations have in general many solutions for \mathbf{a} , each corresponding to a vacuum $|\mathbf{a}\rangle = |0\rangle$ of the quantum gauge theory [1].

2.2 Instantons and localization

The expectation values $\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle$ are analytic functions of the variables a_i . Thus, if we can compute them in an open set in \mathbf{a} -space, then their values for arbitrary \mathbf{a} can be obtained by analytic continuation. In the region

$$|a_i - a_j| \gg |\Lambda| \tag{2.8}$$

the theory is weakly coupled and the path integral (2.3) localizes on instanton configurations,

$$\mathcal{O}(\mathbf{a}, \mathbf{g}, q) = \frac{\sum_{k \geq 0} \int_{X_\infty = \text{diag } \mathbf{a}} dm^{(k)} \mathcal{O}(\mathbf{m}^{(k)}) e^{-S_E}}{\sum_{k \geq 0} \int_{X_\infty = \text{diag } \mathbf{a}} dm^{(k)} e^{-S_E}} = \sum_{k \geq 0} \mathcal{O}^{(k)}(\mathbf{a}, \mathbf{g}) q^k. \tag{2.9}$$

We have denoted by $dm^{(k)}$ the measure on the finite dimensional moduli space of instantons of topological charge k and $\mathcal{O}(\mathbf{m}^{(k)})$ the value of the operator \mathcal{O} for the moduli $\mathbf{m}^{(k)}$. The moduli space integrals are in general ambiguous due to small instanton singularities (see for example the first reference in [6], section VII.2). For example, the expectation values (1.2) are ambiguous for $n \geq 2N$. To lift these ambiguities, we consider the non-commutative deformation of the instanton moduli space. This yields natural definitions for the operators (1.2) at any n [13]. This crucial point will be further discussed in section 4. Note that while turning on the non-commutative deformation $\vartheta \neq 0$ is necessary to define the chiral operators at the non-perturbative level, their expectation values do not depend on ϑ which is a real parameter.

A very important property is that the instanton series always have a non-zero radius of convergence. This shows that $\mathcal{O}(\mathbf{a}, \mathbf{g}, q)$ can be obtained exactly by summing up the

series in (2.9). Of course, computing the moduli space integrals for any values of k is a priori extremely difficult.

The calculation can be drastically simplified by using localization techniques [5]. The idea is that the effective action for the instantons can be written in the form

$$\mathcal{S}_E = Q \cdot \Xi + \Gamma \tag{2.10}$$

with $Q \cdot \Gamma = 0$, for some particular nilpotent linear combination Q of the supercharges. The integrals over the instanton moduli space of Q -closed operators (which include the chiral operators we are interested in) then localize on the solutions to

$$Q \cdot \Xi = 0. \tag{2.11}$$

The fixed points of Q can be found explicitly [5]. They correspond to $U(1)$ non-commutative instantons which, in the commutative limit $\vartheta \rightarrow 0$, go to point-like singular instanton configurations. The remaining integrals over the moduli space of $U(1)$ non-commutative instantons are simpler than the original integrals in (2.9), but their explicit evaluation remains a difficult challenge that has been solved only at topological charges $k \leq 2$.

Very fortunately, it is possible to improve the localization techniques by putting the theory in the so-called Ω -background [2]. This background is characterized by an antisymmetric matrix $\Omega_{\mu\nu}$ that we can choose to be of the form

$$\Omega = \epsilon \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.12}$$

The complex parameter ϵ measures the strength of the background (it is also often denoted by \hbar in the literature). A non-zero Ω -background breaks Lorentz invariance and the usual supersymmetry. For example, the standard kinetic term for the field X is replaced by

$$\text{Tr}(D_\mu X - \Omega_{\nu\lambda} x_\lambda F_{\mu\nu})(D_\mu X^\dagger - \Omega_{\nu\lambda}^\dagger x_\lambda F_{\mu\nu}). \tag{2.13}$$

However, an appropriate deformation of Q , that we denote by Q_ϵ , is preserved, and the action keeps the form (2.10) with ϵ -modified quantities. The truly remarkable fact [2] is that the solutions to the new localization problem associated with Q_ϵ are now labeled by discrete indices. This means that the integrals in (2.9) are reduced to finite sums!

2.3 Colored partitions

Let us describe in details the configurations that contribute [2]. First, a given topological charge k can be distributed amongst the N possible $U(1)$ non-commutative instantons corresponding to the N $U(1)$ factors of the unbroken gauge group (for arbitrary \mathbf{a}),

$$k = \sum_{i=1}^N k_i. \tag{2.14}$$

To each integer $k_i \geq 0$, we associate a partition

$$k_i = \sum_{\alpha \geq 0} k_{i,\alpha}, \tag{2.15}$$

with

$$k_{i,1} \geq k_{i,2} \geq \dots \geq k_{i,\tilde{k}_{i,1}} > k_{i,\tilde{k}_{i,1}+1} = 0. \tag{2.16}$$

The largest integer α such that $k_{i,\alpha} \neq 0$ is denoted by $\tilde{k}_{i,1}$, for reasons to become clear later. A collection of integers $k_{i,\alpha}$ satisfying (2.16) will be symbolically denoted by \mathbf{k}_i and the size of the partition \mathbf{k}_i is defined to be

$$|\mathbf{k}_i| = k_i = \sum_{\alpha=1}^{\tilde{k}_{i,1}} k_{i,\alpha}. \tag{2.17}$$

A colored partition $\vec{\mathbf{k}}$ of size

$$|\vec{\mathbf{k}}| = \sum_{i=1}^N |\mathbf{k}_i| \tag{2.18}$$

is a collection

$$\vec{\mathbf{k}} = (\mathbf{k}_1, \dots, \mathbf{k}_N) \tag{2.19}$$

of N partitions \mathbf{k}_i . The fundamental result [2] is that *the most general instanton configurations that contribute in the topological k sector can be labeled by colored partitions of size $k = |\vec{\mathbf{k}}|$.*

In particular, the partition function Z_ϵ in an arbitrary Ω -background can be written as

$$Z_\epsilon = \sum_{k \geq 0} \int_{X_\infty = \text{diag } \mathbf{a}} dm^{(k)} e^{-S_E} = \sum_{k \geq 0} Z_\epsilon^{(k)} q^k, \tag{2.20}$$

with

$$Z_\epsilon^{(k)} = \sum_{|\vec{\mathbf{k}}|=k} \mu_{\vec{\mathbf{k}}}^2. \tag{2.21}$$

The sum in (2.21) is over all colored partitions of size k , and $\mu_{\vec{\mathbf{k}}}^2$ is a measure factor on the set of colored partitions that we describe below. As the notation suggests, $\mu_{\vec{\mathbf{k}}}^2$ is positive definite when ϵ and the a_i s are chosen to be real. The correlators (2.9) in an arbitrary background are expressed in a similar way,

$$\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle_\epsilon = \mathcal{O}_\epsilon(\mathbf{a}, \mathbf{g}, q, \epsilon) = \frac{1}{Z_\epsilon} \sum_{k \geq 0} q^k \sum_{|\vec{\mathbf{k}}|=k} \mu_{\vec{\mathbf{k}}}^2 \mathcal{O}_{\vec{\mathbf{k}}}, \tag{2.22}$$

where $\mathcal{O}_{\vec{\mathbf{k}}}$ describes the operator \mathcal{O} in the configuration $\vec{\mathbf{k}}$.

It is convenient to introduce the Young tableaux associated with the partitions \mathbf{k}_i in $\vec{\mathbf{k}}$. The Young tableau associated with any partition \mathbf{k} is a collection of boxes arranged in rows, the row number α containing k_α boxes. For example, we have depicted in figure 1 the Young tableau associated with the partition

$$14 = 5 + 3 + 3 + 2 + 1. \tag{2.23}$$

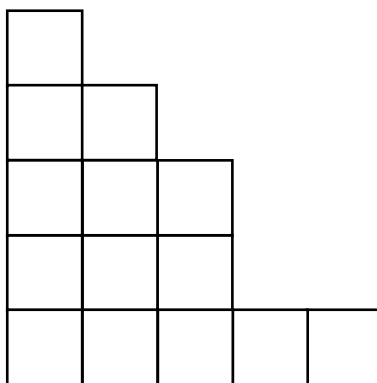


Figure 1: The Young tableau $Y_{\mathbf{k}}$ associated with the partition \mathbf{k} in (2.23), with integers $(k_{\alpha}) = (5, 3, 3, 2, 1)$ and $(\tilde{k}_{\beta}) = (5, 4, 3, 1, 1)$.

In addition to the numbers k_{α} of boxes in the rows, it is useful to also introduce the numbers \tilde{k}_{β} of boxes in the columns, with

$$\tilde{k}_1 \geq \tilde{k}_2 \geq \dots \geq \tilde{k}_{k_1} > \tilde{k}_{k_1+1} = 0. \tag{2.24}$$

The integers \tilde{k}_{β} correspond to the number of boxes in the rows of a partition $\tilde{\mathbf{k}}$ called the dual of \mathbf{k} . Clearly

$$|\mathbf{k}| = \sum_{\alpha=1}^{\tilde{k}_1} k_{\alpha} = \sum_{\beta=1}^{k_1} \tilde{k}_{\beta} = |\tilde{\mathbf{k}}|. \tag{2.25}$$

Let us now consider the box $\square_{(\alpha,\beta)}$ in a tableau $Y_{\mathbf{k}}$ belonging to the row number α and column number β . The Hook length of this box is defined to be

$$h(\square_{(\alpha,\beta)}) = k_{\alpha} - \beta + \tilde{k}_{\beta} - \alpha + 1. \tag{2.26}$$

Geometrically, $h(\square)$ represents the number of boxes above and to the right of \square in the tableau plus one.

We can now give the formula for the measure factor $\mu_{\tilde{\mathbf{k}}}$. Let us start with the case $N = 1$, where only ordinary partitions are involved. Then the measure is simply given in terms of the dimension $\dim R_{\mathbf{k}}$ of the irreducible representation of the symmetric group associated with the Young tableau $Y_{\mathbf{k}}$,

$$\epsilon^{|\mathbf{k}|} \mu_{\mathbf{k}} = \frac{1}{|\mathbf{k}|!} \dim R_{\mathbf{k}}. \tag{2.27}$$

Explicitly,

$$\epsilon^{|\mathbf{k}|} \mu_{\mathbf{k}} = \frac{1}{\prod_{\square \in Y_{\mathbf{k}}} h(\square)}, \tag{2.28}$$

where the product is taken over all the boxes in the Young tableau. For example, for the diagram in figure 1,

$$\epsilon^{14} \mu_{\mathbf{k}} = \frac{1}{9 \cdot 7 \cdot 5 \cdot 2 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \cdot 5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = \frac{1}{1360800}. \tag{2.29}$$

It is possible to write (2.28) is an alternative form which is sometimes useful,

$$\epsilon^{|\mathbf{k}|} \mu_{\mathbf{k}} = \frac{\prod_{1 \leq \alpha_1 < \alpha_2 \leq \tilde{k}_1} (k_{\alpha_1} - k_{\alpha_2} - \alpha_1 + \alpha_2)}{\prod_{\alpha=1}^{\tilde{k}_1} (\tilde{k}_1 + k_{\alpha} - \alpha)!}. \quad (2.30)$$

The equivalence between (2.30) and (2.28) can be shown straightforwardly by using a recursive argument on the number of columns of the Young tableau. A generalization of this result is proven in the appendix. For example, in the case of figure 1, (2.30) yields

$$\epsilon^{14} \mu_{\mathbf{k}} = \frac{3 \cdot 4 \cdot 6 \cdot 8 \cdot 1 \cdot 3 \cdot 5 \cdot 2 \cdot 4 \cdot 2}{9! 6! 5! 3! 1!} = \frac{1}{1360800}, \quad (2.31)$$

consistently with (2.29).

For arbitrary N , the measure is given by a “colored” generalization of (2.28),

$$\begin{aligned} \mu_{\vec{\mathbf{k}}} = & \prod_{i=1}^N \left[\mu_{\mathbf{k}_i} \prod_{\square_{(\alpha,\beta)} \in Y_{\mathbf{k}_i}} \prod_{j \neq i} \frac{1}{a_i - a_j + \epsilon(\beta - \alpha)} \right] \times \\ & \prod_{i < j} \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{j,1}} \frac{(a_i - a_j + \epsilon(\tilde{k}_{j,\beta} - \alpha - \beta + 1))(a_i - a_j + \epsilon(k_{i,\alpha} - \beta - \alpha + 1))}{(a_i - a_j + \epsilon(1 - \alpha - \beta))(a_i - a_j + \epsilon(\tilde{k}_{j,\beta} - \alpha + k_{i,\alpha} - \beta + 1))}. \end{aligned} \quad (2.32)$$

This formula can also be rewritten in a form analogous to (2.30),

$$\begin{aligned} \mu_{\vec{\mathbf{k}}} = & (-1)^{\sum_{i=1}^N (i-1)|\mathbf{k}_i|} \prod_{i=1}^N \mu_{\mathbf{k}_i} \times \prod_{i < j} \left[\prod_{\alpha_1=1}^{\tilde{k}_{i,1}} \prod_{\alpha_2=1}^{\tilde{k}_{j,1}} \frac{a_i - a_j + \epsilon(k_{i,\alpha_1} - k_{j,\alpha_2} - \alpha_1 + \alpha_2)}{a_i - a_j + \epsilon(\alpha_2 - \alpha_1)} \right. \\ & \left. \prod_{\square_{(\alpha,\beta)} \in Y_{\mathbf{k}_i}} \frac{1}{a_i - a_j + \epsilon(\beta - \alpha + \tilde{k}_{j,1})} \prod_{\square_{(\alpha,\beta)} \in Y_{\mathbf{k}_j}} \frac{1}{a_i - a_j - \epsilon(\beta - \alpha + \tilde{k}_{i,1})} \right]. \end{aligned} \quad (2.33)$$

This form has the advantage of making $\mu_{\vec{\mathbf{k}}}^2$ manifestly symmetric under permutation,

$$a_i \leftrightarrow a_j, \quad \mathbf{k}_i \leftrightarrow \mathbf{k}_j, \quad (2.34)$$

which is a consequence of gauge invariance. It is also more convenient to study the $\epsilon \rightarrow 0$ limit. The proof of the equality between (2.32) and (2.33) is given in the appendix.

2.4 The scalar operators

The operators $\text{Tr } X^n$ were studied in [14] for the $\mathcal{N} = 2$ theory. In the configuration $\vec{\mathbf{k}}$, they are given by

$$\begin{aligned} u_{n,\vec{\mathbf{k}}} = & \sum_{i=1}^N \left[a_i^n + \sum_{\alpha=1}^{\tilde{k}_{i,1}} \left((a_i + \epsilon(k_{i,\alpha} - \alpha + 1))^n - (a_i + \epsilon(k_{i,\alpha} - \alpha))^n \right. \right. \\ & \left. \left. + (a_i - \epsilon\alpha)^n - (a_i - \epsilon(\alpha - 1))^n \right) \right]. \end{aligned} \quad (2.35)$$

It is shown in [8], and will be reviewed below, that this formula remains valid in the $\mathcal{N} = 1$ theory as well.

The gauge theory correlators $\langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle$, and thus the quantum superpotential (2.2), can be obtained in principle from the above formulas by taking the $\epsilon \rightarrow 0$ limit,

$$\langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle = \lim_{\epsilon \rightarrow 0} \langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle_{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{1}{Z_{\epsilon}} \sum_{k \geq 0} q^k \sum_{|\vec{k}|=k} \mu_{\vec{k}}^2 u_{n, \vec{k}}. \quad (2.36)$$

This limit was studied in [7] by using the saddle point method. The saddle point corresponds to a very large colored partition, of size $|\vec{k}| \sim 1/\epsilon^2$, for which the shapes of the associated Young tableaux can be computed exactly. The result [7] shows that the generating function is given by

$$R(z; \mathbf{a}, q) = \frac{P'(z)}{\sqrt{P(z)^2 - 4q}}. \quad (2.37)$$

It is a meromorphic function on the Seiberg-Witten curve

$$\mathcal{C} : y^2 = P(z)^2 - 4q = \prod_{i=1}^N (z - x_i)^2 - 4q. \quad (2.38)$$

This curve is a two-sheeted covering of the complex z -plane, with branch cuts running from x_i^- to x_i^+ with

$$P(z) \mp 2q^{1/2} = \prod_{i=1}^N (z - x_i^{\pm}). \quad (2.39)$$

The parameters x_i are determined in terms of the boundary conditions a_j by the equations

$$a_i = \frac{1}{2i\pi} \oint_{\alpha_i} z R(z) dz, \quad (2.40)$$

where the closed contour α_i encircles the cut from x_i^- to x_i^+ .

2.5 Geometric formulation

There is a nice geometric formulation of the localization on the instanton moduli space that uses the notion of equivariant differential forms. Details on this theory can be found for example in [15]. We shall need only a few qualitative features, that were also used in [16, 8]. The idea is that Q_{ϵ} -closed operators correspond to equivariantly closed forms with respect to the symmetry transformation generated by Q_{ϵ} . For our purposes, the important part of this symmetry is a space-time rotation that enters when the Ω -background is turned on. It is generated by the vector field

$$\xi = \Omega_{\mu\nu} x_{\nu} \frac{\partial}{\partial x_{\mu}} = \epsilon \left(iz_1 \frac{\partial}{\partial z_1} - i\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - iz_2 \frac{\partial}{\partial z_2} + i\bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right). \quad (2.41)$$

The complex coordinates z_1 and z_2 are defined by

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4. \quad (2.42)$$

Important equivariant forms (i.e., forms that are invariant under the transformation $z_1 \rightarrow e^{i\gamma} z_1$, $z_2 \rightarrow e^{-i\gamma} z_2$ generated by ξ) on space-time are given by¹

$$\alpha_{(0,0)} = 1 \tag{2.43}$$

$$\alpha_{(2,0)} = dz_1 \wedge dz_2 + i\epsilon z_1 z_2, \tag{2.44}$$

$$\alpha_{(0,2)} = d\bar{z}_1 \wedge d\bar{z}_2 - i\epsilon \bar{z}_1 \bar{z}_2, \tag{2.45}$$

$$\alpha_{(2,2)} = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + i\epsilon(z_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 - \bar{z}_1 \bar{z}_2 dz_1 \wedge dz_2) + \epsilon^2 z_1 z_2 \bar{z}_1 \bar{z}_2. \tag{2.46}$$

It is trivial to check that all these forms are equivariantly closed,

$$(d - i\xi)\alpha_{(n,m)} = 0. \tag{2.47}$$

Equivariantly closed forms on $\mathbb{C}^2 \times \mathcal{M}^{(k)}$, where \mathbb{C}^2 is the space-time and $\mathcal{M}^{(k)}$ the instanton moduli space, can then be built from the equivariant field strength \mathcal{F} [18]. The field strength \mathcal{F} is expressed in terms of the ADHM data; it is a linear combination of the usual Yang-Mills field strength, gluino, scalar and fermion in the chiral multiplet X such that

$$\int d^4x \operatorname{Tr} X^n = \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \operatorname{Tr} \mathcal{F}^n, \tag{2.48}$$

$$\int d^4x \operatorname{Tr} W^\alpha W_\alpha X^n = \frac{16\pi^2}{(n+1)(n+2)} \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \operatorname{Tr} \mathcal{F}^{n+2}, \tag{2.49}$$

$$\int d^4x \int d^2\theta \operatorname{Tr} X^n = \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \operatorname{Tr} \mathcal{F}^n, \tag{2.50}$$

$$\int d^4x \int d^2\theta \operatorname{Tr} W^\alpha W_\alpha X^n = \frac{16\pi^2}{(n+1)(n+2)} \int_{\mathbb{C}^2} \operatorname{Tr} \mathcal{F}^{n+2}. \tag{2.51}$$

Equations (2.48), (2.49) and (2.50) were obtained in [8] (we have simply put the correct factors to match with our conventions). Equation (2.51) can be obtained similarly by a straightforward calculation from the explicit expression for \mathcal{F} .

The integral of an equivariantly closed form localizes on the fixed point of the associated symmetry transformation [15]. All we need is that, for any form α satisfying (2.47),

$$\int_{\mathbb{C}^2} \alpha = \frac{1}{\epsilon^2} \alpha^{(0)}, \tag{2.52}$$

where $\alpha^{(0)}$ is the zero-form part of α evaluated at the origin O of space-time where the vector (2.41) vanishes.² One must be careful in applying this rule because we have regulated the integrals over the instanton moduli space by formulating the theory on a non-commutative space-time. The coordinates z_1 and z_2 are really operators satisfying

$$[\hat{z}_a, \hat{z}_b] = \vartheta \delta_{ab}, \tag{2.53}$$

for which we can use the representation

$$\hat{z}_a = \vartheta \frac{\partial}{\partial \bar{z}_a}. \tag{2.54}$$

¹These forms appear in [8], and we have simply corrected a minus sign.

²We define the integral $\int_{\mathbb{C}^2}$ in such a way that there is no overall constant factor in (2.52).

For example, if we compute the volume of space-time using the form (2.46) and (2.52), we find

$$V = \int_{\mathbb{C}^2} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 = \int_{\mathbb{C}^2} \alpha_{(2,2)} = \frac{1}{\epsilon^2} \epsilon^2 \hat{z}_1 \hat{z}_2 \hat{\bar{z}}_1 \hat{\bar{z}}_2 |_{\mathcal{O}} = \vartheta^2. \quad (2.55)$$

The same calculation for the integral in the right hand side of (2.48) yields

$$\int d^4x \operatorname{Tr} X^n = \vartheta^2 (\operatorname{Tr} \mathcal{F}^n)^{(0)} = \vartheta^2 \operatorname{Tr} X^n, \quad (2.56)$$

showing that the zero-form part of $\operatorname{Tr} \mathcal{F}^n$ is given by

$$(\operatorname{Tr} \mathcal{F}^n)^{(0)} = \operatorname{Tr} X^n. \quad (2.57)$$

This result will be useful later.

Another simple application is to derive the result of [8] that the $\langle \mathbf{a} | \operatorname{Tr} X^n | \mathbf{a} \rangle_\epsilon$ do not depend on \mathbf{g} . We write the euclidean action as a sum of two terms, the $\mathcal{N} = 2$ action that does not depend on the couplings \mathbf{g} and the $\mathcal{N} = 1$ superpotential term,

$$\begin{aligned} \mathcal{S}_E &= \mathcal{S}_{\mathcal{N}=2} + N \int d^4x \int d^2\theta \operatorname{Tr} W(X) + \text{c. c.} \\ &= \mathcal{S}_{\mathcal{N}=2} + N \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \operatorname{Tr} W(\mathcal{F}) + \text{c. c.} \end{aligned} \quad (2.58)$$

The overall factor of N in (2.58) is a natural convention, consistent with (2.1) and (2.2), that makes the action of order N^2 . We have also used (2.50) to rewrite the superpotential term as the integral of an equivariantly closed form. We shall no longer indicate explicitly the anti-chiral terms in the following (the +c. c. in (2.58)), since they obviously do not contribute to the chiral operators expectation values. The idea is now to expand the factor $e^{-\mathcal{S}_E}$ in the path integral in powers of W and then to apply the localization formula (2.52). Since the zero-form part of $\alpha_{(2,0)}$ contains only $z_1 z_2$, a p^{th} power of W yields $(z_1 z_2)^p$. On the other hand, the insertion of $\operatorname{Tr} X^n$ yields, according to (2.48) and (2.46), a factor of $z_1 z_2 \bar{z}_1 \bar{z}_2$. Taking into account the non-commutativity, we have to compute

$$(\hat{z}_1 \hat{z}_2)^{p+1} \hat{\bar{z}}_1 \hat{\bar{z}}_2 |_{\mathcal{O}} = \vartheta^2 \delta_{p,0}, \quad (2.59)$$

showing that there is no dependence in W . The same reasoning also shows that the correlators $\langle \mathbf{a} | \operatorname{Tr} X^{n_1} \cdots \operatorname{Tr} X^{n_s} | \mathbf{a} \rangle_\epsilon$ are independent of \mathbf{g} as well. This is non-trivial because the multi-trace correlators do not factorize at finite ϵ but only in the $\epsilon \rightarrow 0$ limit.

2.6 The glueball operators

Let us now derive the basic formula for the expectation values of the generalized glueball operators,

$$\begin{aligned} -\frac{1}{16\pi^2} \langle \mathbf{a} | \operatorname{Tr} W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon &= \frac{N}{(n+1)(n+2)} \frac{1}{\epsilon^2} \left(\langle \mathbf{a} | \operatorname{Tr} W(X) \operatorname{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon \right. \\ &\quad \left. - \langle \mathbf{a} | \operatorname{Tr} W(X) | \mathbf{a} \rangle_\epsilon \langle \mathbf{a} | \operatorname{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon \right). \end{aligned} \quad (2.60)$$

This formula relates the glueballs to the $\langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle_\epsilon$ computed in 2.4. It appears in the special case of $W(X) = \frac{1}{2}mX^2$ in [8]. Of course, we are mainly interested in the $\epsilon \rightarrow 0$ gauge theory limit

$$v_n(\mathbf{a}, \mathbf{g}, q) = -\frac{1}{16\pi^2} \lim_{\epsilon \rightarrow 0} \langle \mathbf{a} | \text{Tr } W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon. \quad (2.61)$$

A very interesting aspect of (2.60) is to show that the glueball expectation values are related to the *subleading* terms in the small ϵ expansion of $\langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle_\epsilon$. This means that the first corrections in the Ω -background are relevant to the $\mathcal{N} = 1$ gauge theory. In particular, the leading $\epsilon \rightarrow 0$ approximation studied in [7] to solve the $\mathcal{N} = 2$ theory is not sufficient for the case of $\mathcal{N} = 1$.

Equation (2.60) is the main starting point for the calculations performed in sections 3 and 4. We are going to give two derivations. The first one follows closely the reasoning in [8]. The second one uses the properties of the quantum superpotential W_{mic} . A third derivation, which is less formal and completely explicit, will also be given in [12] using an extended version of the theory.

2.6.1 First derivation

Using (2.49), we have

$$-\frac{1}{16\pi^2} \langle \mathbf{a} | \text{Tr } W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon = -\frac{1}{16\pi^2} \int \frac{d^4x}{V} \langle \mathbf{a} | \text{Tr } W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon \quad (2.62)$$

$$= -\frac{1}{(n+1)(n+2)} \frac{1}{\vartheta^2} \langle \mathbf{a} | \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \text{Tr } \mathcal{F}^{n+2} | \mathbf{a} \rangle_\epsilon. \quad (2.63)$$

The zero-form part of $\alpha_{(0,2)}$ in (2.45) is proportional to $\bar{z}_1 \bar{z}_2$. From (2.59), we know that the localization procedure can yield non-zero contributions only if this term is saturated with another contribution in $z_1 z_2$. According to (2.58) and (2.44), such a contribution can come only from a term linear in the superpotential W . This is produced by expanding e^{-S_E} to linear order in W . Using (2.52) and (2.57), we see that the numerator of (2.9) yields a term

$$\begin{aligned} & -\frac{1}{(n+1)(n+2)} \frac{1}{\vartheta^2} \langle \mathbf{a} | -N \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \text{Tr } W(\mathcal{F}) \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \text{Tr } \mathcal{F}^{n+2} | \mathbf{a} \rangle_\epsilon = \\ & \frac{N}{(n+1)(n+2)} \frac{1}{\vartheta^2} \frac{i\epsilon}{\epsilon^2} \frac{-i\epsilon}{\epsilon^2} \vartheta^2 \langle \mathbf{a} | \text{Tr } W(X) \text{Tr } X^{n+2} | \mathbf{a} \rangle_\epsilon = \\ & \frac{N}{(n+1)(n+2)} \frac{1}{\epsilon^2} \langle \mathbf{a} | \text{Tr } W(X) \text{Tr } X^{n+2} | \mathbf{a} \rangle_\epsilon \end{aligned} \quad (2.64)$$

and the denominator of (2.58) yields

$$\begin{aligned} & -\frac{1}{(n+1)(n+2)} \frac{1}{\vartheta^2} \langle \mathbf{a} | N \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \text{Tr } W(\mathcal{F}) | \mathbf{a} \rangle_\epsilon \langle \mathbf{a} | \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \text{Tr } \mathcal{F}^{n+2} | \mathbf{a} \rangle_\epsilon = \\ & -\frac{N}{(n+1)(n+2)} \frac{1}{\vartheta^2} \frac{i\epsilon}{\epsilon^2} \langle \mathbf{a} | \text{Tr } W(X) | \mathbf{a} \rangle_\epsilon \frac{-i\epsilon}{\epsilon^2} \vartheta^2 \langle \mathbf{a} | \text{Tr } X^{n+2} | \mathbf{a} \rangle_\epsilon = \\ & -\frac{N}{(n+1)(n+2)} \frac{1}{\epsilon^2} \langle \mathbf{a} | \text{Tr } W(X) | \mathbf{a} \rangle_\epsilon \langle \mathbf{a} | \text{Tr } X^{n+2} | \mathbf{a} \rangle_\epsilon \end{aligned} \quad (2.65)$$

Combining (2.64) and (2.65) together, we obtain (2.60).

2.6.2 Second derivation

Let us perturb the theory by adding to the tree-level superpotential $\text{Tr} W(X)$ a term $-\frac{t}{16\pi^2} \text{Tr} W^\alpha W_\alpha X^n$. According to (2.51), the new euclidean action is thus

$$\mathcal{S}_E = \mathcal{S}_{\mathcal{N}=2} + N \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \text{Tr} W(\mathcal{F}) - \frac{Nt}{(n+1)(n+2)} \int_{\mathbb{C}^2} \text{Tr} \mathcal{F}^{n+2} + \text{c. c.} \quad (2.66)$$

The formula (2.2) for the quantum superpotential is still valid for non-zero t and ϵ . This follows from the fact that t and ϵ have charge zero under the $U(1)_R$ symmetry (2.4). Moreover, we have, similarly to (2.5) and (2.6),

$$\frac{\partial W_{\text{mic}}}{\partial t} = -\frac{1}{16\pi^2} \langle \mathbf{a} | \text{Tr} W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon = \frac{\partial \langle \mathbf{a} | \text{Tr} W(X) | \mathbf{a} \rangle_\epsilon}{\partial t}. \quad (2.67)$$

Using (2.48), this is equivalent to

$$-\frac{1}{16\pi^2} \langle \mathbf{a} | \text{Tr} W^\alpha W_\alpha X^n | \mathbf{a} \rangle_\epsilon = \frac{1}{\vartheta^2} \frac{\partial}{\partial t} \langle \mathbf{a} | \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \text{Tr} W(\mathcal{F}) | \mathbf{a} \rangle_\epsilon. \quad (2.68)$$

This identity is the starting point of our second derivation of (2.60) (compare with the starting point (2.63) of the first derivation). The use of the localization procedure is particularly simple here, because the zero-form part of $\alpha_{(2,2)}$ is proportional to $z_1 z_2 \bar{z}_1 \bar{z}_2$ and thus non-zero contributions can only come from terms proportional to the trivial form (2.43), i.e. from the term proportional to t in (2.66). The expectation value in (2.68) is given by the general formula (2.9). Taking the derivative of the numerator with respect to t and using (2.66) then yields

$$\begin{aligned} \frac{1}{\vartheta^2} \langle \mathbf{a} | \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \text{Tr} W(\mathcal{F}) \frac{N}{(n+1)(n+2)} \int_{\mathbb{C}^2} \text{Tr} \mathcal{F}^{n+2} | \mathbf{a} \rangle_\epsilon &= \\ \frac{N}{(n+1)(n+2)} \frac{1}{\vartheta^2} \frac{\epsilon^2}{\epsilon^2} \frac{1}{\epsilon^2} \vartheta^2 \langle \mathbf{a} | \text{Tr} W(X) \text{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon &= \\ \frac{N}{(n+1)(n+2)} \frac{1}{\epsilon^2} \langle \mathbf{a} | \text{Tr} W(X) \text{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon, & \quad (2.69) \end{aligned}$$

whereas the variation of the denominator yields

$$\begin{aligned} -\frac{1}{\vartheta^2} \langle \mathbf{a} | \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \text{Tr} W(\mathcal{F}) | \mathbf{a} \rangle \frac{N}{(n+1)(n+2)} \langle \mathbf{a} | \int_{\mathbb{C}^2} \text{Tr} \mathcal{F}^{n+2} | \mathbf{a} \rangle_\epsilon &= \\ -\frac{N}{(n+1)(n+2)} \frac{1}{\vartheta^2} \frac{\epsilon^2}{\epsilon^2} \vartheta^2 \langle \mathbf{a} | \text{Tr} W(X) | \mathbf{a} \rangle_\epsilon \frac{1}{\epsilon^2} \langle \mathbf{a} | \text{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon &= \\ -\frac{N}{(n+1)(n+2)} \frac{1}{\epsilon^2} \langle \mathbf{a} | \text{Tr} W(X) | \mathbf{a} \rangle_\epsilon \langle \mathbf{a} | \text{Tr} X^{n+2} | \mathbf{a} \rangle_\epsilon. & \quad (2.70) \end{aligned}$$

Combining (2.69) and (2.70), we obtain again (2.60) (which is valid for any value of t , even though we are focusing on the $t = 0$ theory).

3. Two instanton calculations at order ϵ^2

3.1 The expectation values $\langle \mathbf{a} | \text{Tr } X^n \text{Tr } X^m | \mathbf{a} \rangle_\epsilon$

In this section, we compute explicitly the correlators $\langle \mathbf{a} | \text{Tr } X^n \text{Tr } X^m | \mathbf{a} \rangle_\epsilon$ up to two instantons,

$$u_{n,m}(\mathbf{a}, q, \epsilon) = \langle \mathbf{a} | \text{Tr } X^n \text{Tr } X^m | \mathbf{a} \rangle_\epsilon = u_{n,m}^{(0)}(\mathbf{a}) + u_{n,m}^{(1)}(\mathbf{a}, \epsilon) q + u_{n,m}^{(2)}(\mathbf{a}, \epsilon) q^2 + \mathcal{O}(q^3). \quad (3.1)$$

Our main goal is to use the resulting formulas to compute the glueball operators (section 3.2) and to check the anomaly equations (section 4). For this purpose, we are particularly interested in the first corrections at small ϵ ,

$$u_{n,m}^{(k)}(\mathbf{a}, \epsilon) = u_{n,m}^{(k,0)}(\mathbf{a}) + u_{n,m}^{(k,2)}(\mathbf{a}) \epsilon^2 + \mathcal{O}(\epsilon^4). \quad (3.2)$$

Note that the functions $u_{n,m}^{(k)}(\mathbf{a}, \epsilon)$ are even in ϵ , to any order. This result is proven in the appendix. Our starting formula, which is a special case of (2.22), is given by

$$u_{n,m}(\mathbf{a}, q, \epsilon) = \frac{1}{Z_\epsilon} \sum_{k \geq 0} q^k \sum_{|\vec{k}|=k} \mu_{\vec{k}}^2 u_{n,\vec{k}} u_{m,\vec{k}}. \quad (3.3)$$

The various ingredients entering into this formula are defined in (2.20), (2.21), (2.32) and (2.35). Expanding at small q both the numerator and the denominator in (3.3), we find that

$$u_{n,m}^{(0)} = u_n^{\text{cl}} u_m^{\text{cl}}, \quad (3.4)$$

$$u_{n,m}^{(1)} = \sum_{|\vec{k}|=1} \mu_{\vec{k}}^2 \left(u_n^{\text{cl}} (u_{m,\vec{k}} - u_m^{\text{cl}}) + u_m^{\text{cl}} (u_{n,\vec{k}} - u_n^{\text{cl}}) + (u_{n,\vec{k}} - u_n^{\text{cl}}) (u_{m,\vec{k}} - u_m^{\text{cl}}) \right), \quad (3.5)$$

$$u_{n,m}^{(2)} = \sum_{|\vec{k}|=2} \mu_{\vec{k}}^2 \left(u_n^{\text{cl}} (u_{m,\vec{k}} - u_m^{\text{cl}}) + u_m^{\text{cl}} (u_{n,\vec{k}} - u_n^{\text{cl}}) + (u_{n,\vec{k}} - u_n^{\text{cl}}) (u_{m,\vec{k}} - u_m^{\text{cl}}) \right) - Z_\epsilon^{(1)} u_{n,m}^{(1)}, \quad (3.6)$$

where we have defined

$$u_n^{\text{cl}} = \sum_{i=1}^N a_i^n. \quad (3.7)$$

One instanton. There are N colored partitions $\vec{k}^{(i)}$ of size $|\vec{k}^{(i)}| = 1$, which describe one instanton in each $U(1)$ factor of the unbroken gauge group, each contributing one term in the sum (3.5). Explicitly,

$$k_{j,\alpha}^{(i)} = \delta_{i,j} \delta_{\alpha,1}, \quad 1 \leq i \leq N, \quad (3.8)$$

and (2.32) or (2.33) then yields

$$\mu_{\vec{k}^{(i)}}^2 = \frac{1}{\epsilon^2} \frac{1}{\prod_{j \neq i} (a_j - a_i)^2}. \quad (3.9)$$

From (2.35) we also get

$$u_{n,\vec{k}^{(i)}} = u_n^{\text{cl}} + \frac{n!}{(n-2)!} a_i^{n-2} \epsilon^2 + \frac{n!}{(n-4)!} \frac{a_i^{n-4}}{12} \epsilon^4 + \frac{n!}{(n-6)!} \frac{a_i^{n-6}}{360} \epsilon^6 + \mathcal{O}(\epsilon^8). \quad (3.10)$$

To express the result, it is convenient to introduce the notation

$$a_{ij} = a_i - a_j. \quad (3.11)$$

Combining (3.9) and (3.10) in (3.5) then yields

$$u_{n,m}^{(1,0)} = \sum_i \frac{1}{\prod_{j \neq i} a_{ij}^2} \left(\frac{m!}{(m-2)!} u_n^{\text{cl}} a_i^{m-2} + \frac{n!}{(n-2)!} u_m^{\text{cl}} a_i^{n-2} \right), \quad (3.12)$$

$$u_{n,m}^{(1,2)} = \sum_i \frac{1}{\prod_{j \neq i} a_{ij}^2} \left(\frac{m!}{12(m-4)!} u_n^{\text{cl}} a_i^{m-4} + \frac{n!}{12(n-4)!} u_m^{\text{cl}} a_i^{n-4} \right. \\ \left. + \frac{n!m!}{(n-2)!(m-2)!} a_i^{n+m-4} \right), \quad (3.13)$$

$$u_{n,m}^{(1,4)} = \sum_i \frac{1}{\prod_{j \neq i} a_{ij}^2} \left(\frac{m!}{360(m-6)!} u_n^{\text{cl}} a_i^{m-6} + \frac{n!}{360(n-6)!} u_m^{\text{cl}} a_i^{n-6} \right. \\ \left. + \frac{n!m!}{12(n-4)!(m-2)!} a_i^{n+m-6} + \frac{n!m!}{12(n-2)!(m-4)!} a_i^{n+m-6} \right). \quad (3.14)$$

Let us note that the term $u_{n,m}^{(1,4)}$, that contributes for one instanton at order ϵ^4 , also contributes at two instantons at order ϵ^2 , and thus will be crucial to get the correct two-instantons correction to the glueball operators. This ϵ^2 contribution comes from the last term in (3.6), taking into account the fact that $Z_\epsilon^{(1)} \propto 1/\epsilon^2$. This is a general feature of these expansions: to get the ϵ^{2q} terms at k -instantons, one needs to compute to order $\epsilon^{2(q+k-k')}$ at $k' < k$ instantons, because $Z_\epsilon^{(k')} \propto 1/\epsilon^{2k'}$.

Two instantons. The sum in (3.6) has $N(N+3)/2$ terms, given by the colored partitions $\vec{k}^{(i)}$ and $\vec{k}^{(ij)}$ characterized by

$$k_{j,\alpha}^{(i)} = \delta_{i,j}(\delta_{\alpha,1} + \delta_{\alpha,2}), \quad 1 \leq i \leq N, \quad (3.15)$$

$$k_{l,\alpha}^{(ij)} = (\delta_{i,l} + \delta_{j,l})\delta_{\alpha,1}, \quad 1 \leq i \leq j \leq N. \quad (3.16)$$

Computing carefully $\mu_{\vec{k}^{(i)}}^2$, $\mu_{\vec{k}^{(ij)}}^2$, $u_{n,\vec{k}^{(i)}}$ and $u_{n,\vec{k}^{(ij)}}$ from (2.32) and (2.35), and plugging into (3.6), we find the following explicit two-instantons result at order ϵ^2 ,

$$u_{n,m}^{(2,0)} = m(m-1) u_n^{\text{cl}} \left[\sum_i \frac{1}{\prod_{l \neq i} a_{il}^4} \left(2 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^2 a_i^{m-2} + \sum_{l \neq i} \frac{1}{a_{il}^2} a_i^{m-2} \right. \right. \\ \left. \left. - (m-2) \sum_{l \neq i} \frac{1}{a_{il}} a_i^{m-3} + \frac{(m-2)(m-3)}{4} a_i^{m-4} \right) + \sum_{i \neq j} \frac{1}{\prod_{l \neq i} a_{il}^2 \prod_{l \neq j} a_{jl}^2} \frac{2a_i^{m-2}}{a_{ij}^2} \right] \\ + (n \leftrightarrow m) + n(n-1)m(m-1) \sum_{i,j} \frac{a_i^{n-2} a_j^{m-2}}{\prod_{l \neq i} a_{il}^2 \prod_{l \neq j} a_{jl}^2}, \quad (3.17)$$

$$\begin{aligned}
 u_{n,m}^{(2,2)} = & m(m-1) u_n^{\text{cl}} \left[\sum_i \frac{1}{\prod_{l \neq i} a_{il}^4} \left(\left[\frac{2}{3} \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^4 + 2 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^2 \sum_{l \neq i} \frac{1}{a_{il}^2} \right. \right. \right. \\
 & + \frac{4}{3} \left(\sum_{l \neq i} \frac{1}{a_{il}} \right) \sum_{l \neq i} \frac{1}{a_{il}^3} + \frac{1}{2} \left(\sum_{l \neq i} \frac{1}{a_{il}^2} \right)^2 + \left. \left. \left. \frac{1}{2} \sum_{l \neq i} \frac{1}{a_{il}^4} \right] a_i^{m-2} \right. \right. \\
 & - \frac{m-2}{2} \left[\frac{4}{3} \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^3 + 2 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right) \sum_{l \neq i} \frac{1}{a_{il}^2} + \frac{2}{3} \sum_{l \neq i} \frac{1}{a_{il}^3} \right] a_i^{m-3} \\
 & + \frac{(m-2)(m-3)}{3} \left[2 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^2 + \sum_{l \neq i} \frac{1}{a_{il}^2} \right] a_i^{m-4} - \frac{(m-2)!}{4(m-5)!} \left(\sum_{l \neq i} \frac{1}{a_{il}} \right) a_i^{m-5} \\
 & \left. + \frac{(m-2)!}{24(m-6)!} a_i^{m-6} \right) + \sum_{i \neq j} \frac{1}{\prod_{l \neq i} a_{il}^2 \prod_{l \neq j} a_{jl}^2} \left(\frac{3 a_i^{m-2}}{a_{ij}^4} + \frac{(m-2)(m-3)}{6} \frac{a_i^{m-4}}{a_{ij}^2} \right) \Big] + \\
 & + (n \leftrightarrow m) + n(n-1)m(m-1) \left[\sum_i \frac{1}{\prod_{l \neq i} a_{il}^4} \left(\frac{(n-2)(m-2)}{2} a_i^{n+m-6} \right. \right. \\
 & + \frac{7}{12} \left((n-2)(n-3) + (m-2)(m-3) \right) a_i^{n+m-6} - 2(n+m-4) \left(\sum_{l \neq i} \frac{1}{a_{il}} \right) a_i^{n+m-5} \\
 & \left. \left. + 4 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^2 a_i^{n+m-4} + 2 \left(\sum_{l \neq i} \frac{1}{a_{il}^2} \right) a_i^{n+m-4} \right) \right. \\
 & \left. + \sum_{i \neq j} \frac{1}{\prod_{l \neq i} a_{il}^2 \prod_{l \neq j} a_{jl}^2} \left(\frac{(n-2)(n-3)}{12} a_i^{n-4} a_j^{m-2} + \frac{(m-2)(m-3)}{12} a_i^{m-4} a_j^{n-2} \right. \right. \\
 & \left. \left. + \frac{2}{a_{ij}^2} \left(a_i^{n+m-4} + a_i^{n-2} a_j^{m-2} \right) \right) \right]. \quad (3.18)
 \end{aligned}$$

Let us note that as a special case of the above calculation, we also find the expectation values of $\langle \mathbf{a} | \text{Tr } X^n | \mathbf{a} \rangle_\epsilon$,

$$u_n(\mathbf{a}, q, \epsilon) = \frac{u_{n,0}(\mathbf{a}, q, \epsilon)}{N} = u_n^{\text{cl}}(\mathbf{a}) + u_n^{(1)}(\mathbf{a}, \epsilon) q + u_n^{(2)}(\mathbf{a}, \epsilon) q^2 + \mathcal{O}(q^3), \quad (3.19)$$

and in particular the microscopic quantum superpotential (2.2) is known up to two instants.

3.2 The glueball operators expectation values

We can now use the fundamental formula (2.60) to get the glueball operators expectation values, at $\epsilon = 0$, from the results of the previous subsection. Expanding

$$v_m(\mathbf{a}, \mathbf{g}, q) = v_m^{(1)}(\mathbf{a}, \mathbf{g}) q + v_m^{(2)}(\mathbf{a}, \mathbf{g}) q^2 + \mathcal{O}(q^3), \quad (3.20)$$

we find

$$\begin{aligned}
 v_m^{(1)}(\mathbf{a}, \mathbf{g}) = & \frac{N}{(m+1)(m+2)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \sum_{n \geq 0} \left[\frac{g_n}{n+1} \left(u_{n+1, m+2}^{(1)}(\mathbf{a}, \epsilon) \right. \right. \\
 & \left. \left. - u_{n+1}^{\text{cl}}(\mathbf{a}) u_{m+2}^{(1)}(\mathbf{a}, \epsilon) - u_{m+2}^{\text{cl}}(\mathbf{a}) u_{n+1}^{(1)}(\mathbf{a}, \epsilon) \right) \right], \quad (3.21)
 \end{aligned}$$

$$\begin{aligned}
 v_m^{(2)}(\mathbf{a}, \mathbf{g}) = & \frac{N}{(m+1)(m+2)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \sum_{n \geq 0} \left[\frac{g_n}{n+1} \left(u_{n+1, m+2}^{(2)}(\mathbf{a}, \epsilon) \right. \right. \\
 & \left. \left. - u_{n+1}^{\text{cl}}(\mathbf{a}) u_{m+2}^{(2)}(\mathbf{a}, \epsilon) - u_{m+2}^{\text{cl}}(\mathbf{a}) u_{n+1}^{(2)}(\mathbf{a}, \epsilon) - u_{n+1}^{(1)}(\mathbf{a}, \epsilon) u_{m+2}^{(1)}(\mathbf{a}, \epsilon) \right) \right]. \quad (3.22)
 \end{aligned}$$

A careful calculation then yields the following explicit formulas, for the one-instanton contribution,

$$v_m^{(1)}(\mathbf{a}, \mathbf{g}) = N \sum_i \frac{W''(a_i) a_i^m}{\prod_{l \neq i} a_{il}^2} \quad (3.23)$$

and for the two-instantons contribution,

$$\begin{aligned}
 v_m^{(2)}(\mathbf{a}, \mathbf{g}) = & N \left[\sum_i \frac{1}{\prod_{l \neq i} a_{il}^4} \left(\left[\frac{1}{2} W''''(a_i) - 2 \sum_{l \neq i} \frac{1}{a_{il}} W'''(a_i) + 4 \left(\sum_{l \neq i} \frac{1}{a_{il}} \right)^2 W''(a_i) \right. \right. \right. \\
 & \left. \left. + 2 \sum_{l \neq i} \frac{1}{a_{il}^2} W''(a_i) \right] a_i^m + m \left[\frac{1}{2} W''''(a_i) - 2 \sum_{l \neq i} \frac{1}{a_{il}} W''(a_i) \right] a_i^{m-1} \right. \\
 & \left. + \frac{m(m-1)}{2} W''(a_i) a_i^{m-2} \right) + \sum_{i \neq j} \frac{W''(a_i) + W''(a_j)}{\prod_{l \neq i} a_{il}^2 \prod_{l \neq j} a_{jl}^2} \frac{2a_i^m}{a_{ij}^2} \right]. \quad (3.24)
 \end{aligned}$$

We now have all the necessary ingredients to perform the check of the Dijkgraaf-Vafa matrix model from our purely microscopic point of view. In principle, all we have to do is to show that the above correlators satisfy the generalized Konishi anomaly equations when we go on-shell, i.e. when we extremize W_{mic} (of course the correlators will *not* satisfy the anomaly equations for arbitrary values of \mathbf{a}). We are going to perform this check in the next section, and also exhibit highly non-trivial features of the anomaly equations at the non-perturbative level.

4. Non-perturbative anomaly equations

4.1 Introduction

A cornerstone of our understanding of $\mathcal{N} = 1$ gauge theories, and their relation with the Dijkgraaf-Vafa matrix model, is the set of generalized anomaly equations studied in [10]. These equations have been derived in perturbation theory (i.e. in a fixed classical background gauge field) in the following way [10].

We consider some particular non-linear variations of the field X in the path integral [10], which are generated by the operators

$$L_n = -X^{n+1} \frac{\delta}{\delta X}, \quad J_n = \frac{1}{16\pi^2} W^\alpha W_\alpha X^{n+1} \frac{\delta}{\delta X}, \quad \text{for } n \geq -1. \quad (4.1)$$

In [10] the operators $W^\alpha X^{n+1} \delta / \delta X$ were also considered, but the resulting equations do not produce non-trivial constraints on expectation values.³ The operators act on the gauge

³We could include them straightforwardly in the discussion by introducing Lorentz-violating couplings $t_n^\alpha \text{Tr } W_\alpha X^{n+1}$ in the tree-level superpotential.

invariant observables as

$$L_n \cdot u_m = -m u_{n+m}, \quad J_n \cdot u_m = -m v_{n+m}, \quad L_n \cdot v_m = -m v_{n+m}, \quad J_n \cdot v_m = 0, \quad (4.2)$$

and satisfy the algebra

$$[L_n, L_m] = (n - m)L_{n+m}, \quad [L_n, J_m] = (n - m)J_{n+m}, \quad [J_n, J_m] = 0. \quad (4.3)$$

The relations $J_n \cdot v_m = 0$ and $[J_n, J_m] = 0$ follow from the fact that the W^α anticommutes in the chiral ring. The anomaly polynomials generated by L_n and J_n are respectively [10]

$$\mathcal{A}_n = -N \sum_{k \geq 0} g_k u_{n+k+1} + 2 \sum_{k_1+k_2=n} u_{k_1} v_{k_2}, \quad (4.4)$$

$$\mathcal{B}_n = -N \sum_{k \geq 0} g_k v_{n+k+1} + \sum_{k_1+k_2=n} v_{k_1} v_{k_2}. \quad (4.5)$$

The terms linear in the fields in (4.4) and (4.5) come from the tree-level action, whereas the quadratic terms are generated by an anomalous jacobian in the path integral measure (in the Fujikawa approach) or equivalently by a one-loop calculation with external gauge fields. It is not difficult to show that this result is exact in perturbation theory, to any loop order, for example by using the Wess-Zumino consistency conditions

$$L_n \cdot \mathcal{A}_m - L_m \cdot \mathcal{A}_n = (n - m)\mathcal{A}_{n+m} \quad (4.6)$$

$$L_n \cdot \mathcal{B}_m - J_m \cdot \mathcal{A}_n = (n - m)\mathcal{B}_{n+m} \quad (4.7)$$

$$J_n \cdot \mathcal{B}_m - J_m \cdot \mathcal{B}_n = 0 \quad (4.8)$$

associated with the algebra (4.3).

It is convenient to use operator-valued generating functions for the L_n and J_n ,

$$L(z) = \sum_{n \geq -1} \frac{L_n}{z^{n+2}}, \quad J(z) = \sum_{n \geq -1} \frac{J_n}{z^{n+2}}. \quad (4.9)$$

These operators generate anomaly polynomials that can be written elegantly in terms of the generating functions R and S for the u_n s and v_n s,

$$\mathcal{A}(z) = \sum_{n \geq -1} \frac{\mathcal{A}_n}{z^{n+2}} = -NW'(z)R(z) + 2R(z)S(z) + N^2\Delta_R(z), \quad (4.10)$$

$$\mathcal{B}(z) = \sum_{n \geq -1} \frac{\mathcal{B}_n}{z^{n+2}} = -NW'(z)S(z) + S(z)^2 + N^2\Delta_S(z), \quad (4.11)$$

where Δ_R and Δ_S are polynomials chosen to cancel the terms of positive powers in z in the right-hand sides of (4.10) and (4.11).

4.2 Non-perturbative subtleties and finite N

4.2.1 The non-perturbative anomaly conjecture

The anomaly polynomials (4.4) and (4.5) must vanish on-shell. The resulting equations are very similar to the planar loop equations of the one-matrix model, and this hints at

the formulation in terms of the matrix model in [9]. However, there is a very important difference with the matrix model, that has been overlooked in most of the literature, but which was emphasized in [13]. In the gauge theory, the number of colors N is *finite*, and thus the variables that enter in (4.4) and (4.5) are not independent. Actually, only u_1, \dots, u_N and v_0, \dots, v_{N-1} can be independent, all the other observables being expressed as polynomials in these basic variables. For example, because X is a $N \times N$ matrix, we have

$$u_{N+p} = \mathcal{P}_{\text{cl},p}(u_1, \dots, u_N), \quad p \geq 1, \quad (4.12)$$

for some homogeneous polynomials \mathcal{P}_p of degree $N + p$ (u_n being of degree n) that can be easily computed. It is straightforward to check that the vanishing of the anomaly polynomials can be consistent with (4.12) only if the expectation values do not get quantum corrections at all, providing a proof of the standard perturbative non-renormalization theorem.

These remarks clearly show that the anomaly polynomials must get non-perturbative corrections to be consistent with the non-trivial non-perturbative corrections to the chiral operators expectation values [13]. The precise conjecture about the anomaly equations can then be stated as follows [13]:

Non-perturbative anomaly conjecture. *The non-perturbative corrections to (4.4) and (4.5) are such that they can be absorbed in a non-perturbative redefinition of the variables that enter the equations.*

This means that, at the expense of *defining* the variables u_n and v_{n-1} for $n > N$ in a suitable way, we can assume that the anomaly polynomials (4.4) and (4.5) are exact at the non-perturbative level. The only constraints on the possible definitions of the variables come from the classical limit and the symmetries of the theory, the $U(1)_R$ symmetry (2.4) as well as the $U(1)_A$ symmetry for which the relevant charges are given by

$$\begin{array}{cccc} & u_n & v_n & g_k & q \\ U(1)_R & 0 & 2 & 2 & 0 \\ U(1)_A & n & n & -k - 1 & 2N. \end{array} \quad (4.13)$$

For example, the u_{N+p} that enter in the anomaly polynomials could be given by any formula of the form

$$u_{N+p} = \mathcal{P}_p(u_1, \dots, u_N; q), \quad p \geq 1, \quad (4.14)$$

for polynomials \mathcal{P}_p of $U(1)_A$ charge $N + p$ that goes to $\mathcal{P}_{\text{cl},p}$ when q goes to zero. The precise form of the polynomials \mathcal{P}_p are unknown a priori. However, a little thinking shows that it is actually quite miraculous that the vanishing of the anomaly polynomials can be consistent at all with the existence of non-trivial quantum corrections and relations like (4.14). It was then conjectured in [13] that the form of the polynomials were actually fixed uniquely by consistency with the anomaly equations, and that this requirement was actually equivalent to the extremization of the Dijkgraaf-Vafa superpotential. This conjecture can be proven, including when flavors are added to the theory [19].

In a given non-perturbative microscopic setting, where all the operators u_n and v_n are well-defined, the relations like (4.14) must be fixed. Let us emphasize again that these

relations are mere definitions of what we mean by u_{N+p} for $p \geq 1$, and thus have no dynamical content. In particular, they must be valid off-shell. In our framework, based on the non-commutative regularization of the instanton moduli space, we thus expect to find some explicit form for the polynomials \mathcal{P}_p , with relations (4.14) valid for any values of the boundary conditions \mathbf{a} . This can be easily checked as follows [13].

Let us introduce the correlator

$$F(z; \mathbf{a}, q) = \langle \mathbf{a} | \det(z - X) | \mathbf{a} \rangle. \tag{4.15}$$

We have

$$\frac{F'(z)}{F(z)} = R(z), \tag{4.16}$$

and Nekrasov's formula (2.37) then implies that

$$F(z; \mathbf{a}, q) = \frac{1}{2} \left(P(z) + \sqrt{P(z)^2 - 4q} \right). \tag{4.17}$$

The function F is thus a well-defined meromorphic function on the curve (2.38), and in particular it satisfies an algebraic equation that can be conveniently written in the form

$$F(z) + \frac{q}{F(z)} = P(z). \tag{4.18}$$

Expanding at large z , using the fact that

$$F(z) = z^N e^{-\sum_{n \geq 1} u_n / (nz^n)} \tag{4.19}$$

and that all the terms with negative powers of z in the left hand side of (4.18) must vanish, we obtain an infinite set of equations that generate recursively *and are equivalent to* a specific form for the relations (4.14). For example, we find that

$$\mathcal{P}_p = \mathcal{P}_{\text{cl},p} \quad \text{for } 1 \leq p \leq N-1, \quad \mathcal{P}_N = \mathcal{P}_{\text{cl},N} + 2Nq, \quad \text{etc} \dots \tag{4.20}$$

This is equivalent to saying that the equation (4.18) is not dynamical but simply encodes the off-shell kinematical relations (4.14) (only the explicit form of the polynomial P is dynamical). It is extremely tempting to believe that this natural definition of the operators is precisely the one for which the anomaly equations take the simple forms (4.4) and (4.5). This is suggested by all the known results on the theory, and we will check it explicitly up to two instantons below and to all orders in [12]. However, having non-trivial q -dependent relations like (4.20) between the operators imply some very drastic consequences on the generators L_n and J_n that were defined in perturbation theory by (4.1) or equivalently by (4.2), as we are now going to discuss.

4.2.2 On the quantum corrected operators L_n and J_n

At the non-perturbative level, the operators L_n and J_n clearly can get quantum corrections for $n \geq 1$ because the associated transformations are non-linear. This is a well-known field

theoretic effect, that plays a rôle in many instances, for example in the BRST renormalization theory of Yang-Mills: non-linear transformation rules can be renormalized. Here we are dealing with a particularly interesting non-perturbative example of this effect.

An obvious question to ask is what kind of quantum corrections can modify the operators L_n and J_n and their algebra. This is important for example if one wish to study the possible non-perturbative corrections to the anomaly equations by using the Wess-Zumino consistency conditions, as suggested in [10]. A natural, albeit naïve, guess is that the corrections are mild enough for the operators to remain derivations acting in a closed form on the chiral ring. For example, focusing on the operators L_n and variables u_m , we might assume that in the full quantum theory the most general possibility is to have relations like

$$L_n \cdot u_m = -m u_{n+m} + \sum_{k \geq 1} q^k r_{n,m}^{(k)} \tag{4.21}$$

and

$$[L_n, L_m] = (n - m)L_{n+m} + \sum_{k \geq 1} q^k L_{n,m}^{(k)}, \tag{4.22}$$

where the $r_{n,m}^{(k)}$ are polynomials in the u_p s and the $L_{n,m}^{(k)}$ are operators of A-charges $n + m - 2Nk$, consistently with (4.13). Note that the constraints on the A-charges imply that the instanton series in (4.21) and (4.22) have only a finite number of terms. Constraints like (4.21) are at the basis of the analysis in [20] for example.⁴ However, and perhaps surprisingly, it turns out that the non-perturbative quantum corrections to the operators L_n and J_n must be much stronger. Actually, the formulas (4.21) and (4.22) are *inconsistent* with the existence of the quantum corrected relations (4.14)!

The precise statement is as follows:

Assume that the anomaly equations are given by (4.4) and (4.5) with the u_{N+p} variables defined by (4.14), where the polynomials \mathcal{P}_p are deduced from (4.18).⁵ Assume that relations like (4.21) and (4.22) are also valid. Then necessarily $q = 0$, i.e. the theory is classical.

Let us derive this result in the simple case $N = 2$. We have also done the analysis in the general case, but it is quite tedious and not necessary for our purposes. It will be enough to consider a tree-level superpotential of the form $W(z) = \frac{1}{2}mz^2$. From (4.21) and (4.22), we only need the facts that the $L_n \cdot u_m$ and $[L_n, L_m]$ (and thus the associated Wess-Zumino consistency conditions) are not corrected if $n + m < 4$, as well as

$$L_0 \cdot u_4 = -4u_4 + c_1q, \tag{4.23}$$

$$L_2 \cdot u_2 = -2u_4 + c_2q, \tag{4.24}$$

$$L_1 \cdot u_3 = -3u_4 + c_3q, \tag{4.25}$$

⁴Several assumptions and derivations in [20] are inconsistent and we do not agree with most of the statements in this paper.

⁵These are the standard claims about the theory, and we shall be able to provide a full microscopic derivation below and in [12].

for some numerical constants c_1 , c_2 and c_3 . These constants are not independent. From $[L_2, L_0] = 2L_2$, we deduce

$$L_2 \cdot u_2 = \frac{1}{2}[L_2, L_0] \cdot u_2 = \frac{1}{2}L_2 \cdot (-2u_2) - \frac{1}{2}L_0 \cdot (-2u_4) = -L_2 \cdot u_2 + L_0 \cdot u_4, \quad (4.26)$$

which implies that

$$c_2 = \frac{c_1}{2}. \quad (4.27)$$

Similarly, $[L_2, L_1] = L_3$ acting on u_1 yields

$$L_1 \cdot u_3 = L_3 \cdot u_1 + L_2 \cdot u_2, \quad (4.28)$$

and $[L_3, L_0] = 3L_3$ acting on u_1 yields, by using (4.28),

$$L_1 \cdot u_3 = L_2 \cdot u_2 + \frac{1}{4}L_0 \cdot u_4. \quad (4.29)$$

From (4.23), (4.24) and (4.27) we thus get

$$c_3 = \frac{3c_1}{4}. \quad (4.30)$$

Let us now use the Wess-Zumino consistency conditions (4.6) for $(n, m) = (2, 0)$. Using the explicit formulas

$$\mathcal{A}_0 = -2mu_2 + 4v_0, \quad \mathcal{A}_2 = -2mu_4 + 4v_2 + 2u_1v_1 + 2u_2v_0 \quad (4.31)$$

and (4.23) and (4.24), a direct calculation shows that

$$L_2 \cdot \mathcal{A}_0 - L_0 \cdot \mathcal{A}_2 - 2\mathcal{A}_2 = 2m(c_1 - c_2)q = 0. \quad (4.32)$$

Using (4.27) and (4.30), we deduce that

$$c_1 = c_2 = c_3 = 0. \quad (4.33)$$

Let us now use (4.20) in the cases $N = 2$, $p = 1$ and $p = 2$,

$$u_3 = \mathcal{P}_{\text{cl},1}(u_1, u_2) = \frac{3}{2}u_1u_2 - \frac{1}{2}u_1^3, \quad (4.34)$$

$$u_4 = \mathcal{P}_{\text{cl},2}(u_1, u_2) + 4q = u_1u_3 + \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2u_2 + 4q. \quad (4.35)$$

Acting on (4.34) with the operator L_1 , and using (4.33), yields

$$L_1 \cdot u_3 = -3u_4 = L_1 \cdot \left(\frac{3}{2}u_1u_2 - \frac{1}{2}u_1^3 \right) = -\frac{3}{2}u_2^2 - 3u_1u_3 + \frac{3}{2}u_1^2u_2. \quad (4.36)$$

This is consistent with (4.35) only for $q = 0$, as was to be shown.

4.3 Non-perturbative generators and algebra

We have seen in the previous subsection that the quantum corrections to the generators of the anomaly equations must be very strong, and in particular must violate ansatz like (4.21) and (4.22). It is then very difficult to guess the general form of the allowed corrections a priori. In particular, it seems extremely difficult to try to derive the non-perturbative anomaly conjecture by using the Wess-Zumino consistency conditions.

On the other hand, in the microscopic framework of the present paper, it should be possible in principle to provide a full derivation of the anomaly equations and associated generators and algebra. In our framework, we are thus seeking differential operators L_n and J_n , or more conveniently the generating functions $L(z)$ and $J(z)$ defined in (4.9), that act on the microscopic off-shell variables a_i ,

$$L(z) = \sum_{i=1}^N \delta_z^L a_i \frac{\partial}{\partial a_i}, \quad J(z) = \sum_{i=1}^N \delta_z^J a_i \frac{\partial}{\partial a_i}, \quad (4.37)$$

and such that

$$\begin{aligned} NL(z) \cdot W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) &= \mathcal{A}(z; \mathbf{a}, \mathbf{g}, q) \\ &= -NW'(z)R(z; \mathbf{a}, q) + 2R(z; \mathbf{a}, q)S(z; \mathbf{a}, \mathbf{g}, q), \end{aligned} \quad (4.38)$$

$$\begin{aligned} NJ(z) \cdot W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) &= \mathcal{B}(z; \mathbf{a}, \mathbf{g}, q) \\ &= -NW'(z)S(z; \mathbf{a}, \mathbf{g}, q) + S(z; \mathbf{a}, \mathbf{g}, q)^2. \end{aligned} \quad (4.39)$$

The functions $R(z; \mathbf{a}, q)$ and $S(z; \mathbf{a}, \mathbf{g}, q)$ have been studied extensively in sections 2 and 3. R is explicitly known from the results of [7], see equation (2.37). On the other hand, S can in principle be obtained by summing over colored partitions from (2.60), but we only know its explicit form up to two instantons from the calculations of section 3.

There is a very natural proposal for the operators $L(z)$ and $J(z)$. We conjecture that

$$\delta_z^L a_i = \frac{1}{2i\pi} \oint_{\alpha_i} \frac{R(z'; \mathbf{a}, q)}{z' - z} dz', \quad (4.40)$$

$$\delta_z^J a_i = \frac{1}{2i\pi} \oint_{\alpha_i} \frac{S(z'; \mathbf{a}, \mathbf{g}, q)}{z' - z} dz'. \quad (4.41)$$

In these formulas, the point z is chosen to be outside the contours α_i that were defined in section 2.4. For the L_n and J_n , the corresponding explicit formulas read

$$L_n = -\frac{1}{2i\pi} \sum_{i=1}^N \oint_{\alpha_i} z^{n+1} R(z; \mathbf{a}, q) dz \frac{\partial}{\partial a_i}, \quad (4.42)$$

$$J_n = -\frac{1}{2i\pi} \sum_{i=1}^N \oint_{\alpha_i} z^{n+1} S(z; \mathbf{a}, \mathbf{g}, q) dz \frac{\partial}{\partial a_i}. \quad (4.43)$$

We would like to make two comments on the above formulas.

First, it is not obvious a priori that the formulas for $J(z)$ or J_n make sense, because we do not know if $S(z)$ is a well-defined function on the curve (2.38). Actually, since the

contours α_i lie entirely on the first sheet of the surface, which is defined by the asymptotic conditions

$$R(z; \mathbf{a}, q) \underset{z \rightarrow \infty}{\sim} \frac{N}{z}, \quad S(z; \mathbf{a}, \mathbf{g}, q) \underset{z \rightarrow \infty}{\sim} \frac{v_0(\mathbf{a}, \mathbf{g}, q)}{z}, \quad (4.44)$$

all we need is that $S(z)$ is well defined on this first sheet, with the same branch cuts as $R(z)$. In particular, the conditions

$$\oint_{\alpha_i} S'(z; \mathbf{a}, \mathbf{g}, q) dz = 0 \quad (4.45)$$

must be satisfied. Anticipating a bit the results derived in [12], it can be shown that $S'(z)$ is a well-defined meromorphic function on (2.38) satisfying (4.45), ensuring that the formulas (4.41) and (4.43) do make sense. However, it turns out that the function $S(z)$ itself is *not* well defined on (2.38).

The second comment we would like to make is related to the discussion in section 4.2.2. It is actually quite obvious that a formula like (4.42) must violate (4.21) (with similar statements for the J_n). The reason is that $L_n \cdot u_m(\mathbf{a}, q)$ will in general be a well-defined function of the a_i , but a *multi-valued* function of the u_p . This is the consequence of the well-known non-trivial monodromies that the variables a_i undergo in the u_p -space. Similarly, the algebra of the operators L_n and J_n defined by (4.42) and (4.43) is not closed. This can be checked straightforwardly from (2.37) and the formulas in section 4.1 of [1]. In order to obtain a closed algebra, we need to enlarge the set of operators considerably. Let us see how this work in the case of the operators L_n . We set, for any meromorphic one-form ω on (2.38),

$$\sigma_i(\omega) = \frac{1}{2i\pi} \oint_{\alpha_i} \omega, \quad (4.46)$$

and associate to ω the differential operator defined by

$$L(\omega) = \sum_{i=1}^N \sigma_i(\omega) \frac{\partial}{\partial a_i}. \quad (4.47)$$

The operators L_n are of this form,

$$L_n = L(\omega_n), \quad \omega_n = -z^{n+1} R(z) dz. \quad (4.48)$$

The commutator of two operators $L(\omega)$ and $L(\eta)$ is given in terms of the skew product

$$\langle \omega, \eta \rangle = \sum_{i=1}^N \left(\sigma_i(\omega) \frac{\partial \eta}{\partial a_i} - \sigma_i(\eta) \frac{\partial \omega}{\partial a_i} \right) \quad (4.49)$$

by

$$[L(\omega), L(\eta)] = L(\langle \omega, \eta \rangle). \quad (4.50)$$

Taking the derivative of forms with respect to a_i can introduce poles at the branching points x_i^\pm of the curve (2.38). For this reason, the commutators of the L_n , and then the commutators of commutators, etc, will generate operators $L(\omega)$ with forms ω having poles of higher and higher orders at the branching points x_i^\pm . The resulting infinite dimensional algebra is quite interesting and would deserve further study. In the limit $q \rightarrow 0$ it has the partial Virasoro algebra as a closed subalgebra.

4.4 Checks in the instanton expansion

4.4.1 The anomaly equations

Let us now check explicitly (4.38) and (4.39) by using the results of section 3. The calculation is straightforward, but quite tedious. Actually, finding the correct anomaly polynomials look like a little miracle in the present formalism. This is very unlike the case of the matrix model approach, where the anomaly equations are the most natural identities, and follow directly from the properties of the matrix integral. In the present microscopic formalism based on the sum over colored partitions, we do not have such a simple interpretation.

We have performed all our calculations at the *two*-instantons order. However, the intermediate formulas are so complicated that we are simply going to indicate the main steps, writing explicitly only the terms relevant to the one-instanton order.

First, we write the generating functions explicitly using the formulas derived in section 3,

$$R(z; \mathbf{a}, q) = \sum_i \frac{1}{z - a_i} + 2q \sum_i \frac{1}{\prod_{l \neq i} a_{il}^2} \frac{1}{(z - a_i)^3} + \mathcal{O}(q^2), \quad (4.51)$$

$$S(z; \mathbf{a}, \mathbf{g}, q) = Nq \sum_i \frac{W''(a_i)}{\prod_{l \neq i} a_{il}^2} \frac{1}{z - a_i} + \mathcal{O}(q^2). \quad (4.52)$$

We see that in the small q expansion, the functions R and S are meromorphic functions on the complex plane with poles at the points $z = a_i$. This feature is maintained at any finite order in q , with poles of higher and higher orders as the instanton number increases. The α_i -periods of differential forms involving R and S thus reduce to a sum over the residues at a_i . Using (4.42) and (4.43), we can get in this way the explicit formulas for the operators L_n and J_n ,

$$L_n = - \sum_i \left(a_i^{n+1} + q(n+1)n \frac{a_i^{n-1}}{\prod_{l \neq i} a_{il}^2} \right) \frac{\partial}{\partial a_i} + \mathcal{O}(q^2), \quad (4.53)$$

$$J_n = -Nq \sum_i \frac{W''(a_i) a_i^{n+1}}{\prod_{l \neq i} a_{il}^2} \frac{\partial}{\partial a_i} + \mathcal{O}(q^2). \quad (4.54)$$

We need next to compute $\partial W_{\text{mic}} / \partial a_i$. From (2.2) we know that

$$W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) = \sum_{m \geq 0} \frac{g_m}{m+1} \left(u_{m+1}^{\text{cl}}(\mathbf{a}) + u_{m+1}^{(1,0)}(\mathbf{a}) q + \mathcal{O}(q^2) \right), \quad (4.55)$$

from which we find, using (3.19) and (3.12),

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = W'(a_i) + q \left[\frac{1}{\prod_{l \neq i} a_{il}^2} \left(W'''(a_i) - 2 \sum_{l \neq i} \frac{1}{a_{il}} W''(a_i) \right) - 2 \sum_{j \neq i} \frac{W''(a_j)}{\prod_{l \neq j} a_{jl}^2} \frac{1}{a_{ij}} \right] + \mathcal{O}(q^2). \quad (4.56)$$

Combining (4.56) with (4.53) and (4.54), we can then check explicitly that

$$NL_n \cdot W_{\text{mic}} = \mathcal{A}_n + \mathcal{O}(q^2), \quad NJ_n \cdot W_{\text{mic}} = \mathcal{B}_n + \mathcal{O}(q^2). \quad (4.57)$$

Repeating the same calculation, but now including all the relevant two-instantons terms, we have actually explicitly checked, at the cost of considerable algebra, that

$$NL_n \cdot W_{\text{mic}} = \mathcal{A}_n + \mathcal{O}(q^3), \quad NJ_n \cdot W_{\text{mic}} = \mathcal{B}_n + \mathcal{O}(q^3), \quad (4.58)$$

or equivalently that (4.38) and (4.39) are valid up to terms of order q^3 .

Note that the above results immediately imply that the microscopic approach match the Dijkgraaf-Vafa approach, at least up to two instantons. Indeed, when the equations (2.7) are satisfied, we automatically get

$$NL(z) \cdot W_{\text{mic}} = 0 = \mathcal{A}(z), \quad NJ(z) \cdot W_{\text{mic}} = 0 = \mathcal{B}(z). \quad (4.59)$$

In the Dijkgraaf-Vafa formalism, these equations must be supplemented by the extremization of the glueball superpotential. However, it is well-known (see for example [21, 13]) that this is equivalent to the fact that the quantum characteristic function (4.15) satisfies the algebraic equation (4.18). This latter equation is automatically implemented in the microscopic approach.

There is, of course, a limitation in working at a finite order in the instanton expansion. The equations of motion (2.7) then allow to study only the Coulomb vacuum of the theory, in which the unbroken gauge group has only U(1) factors. This limitation will be waived in [12], using the results of [1], by providing an exact analysis independent of the small q approximation.

4.4.2 The algebra

Let us now compute the first non-trivial quantum corrections to the perturbative algebra (4.3). From (4.53) we find

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{2q}{\prod_{l \neq i} a_{il}^2} \left(n(n+1) a_i^{n-1} \sum_j \sum_{q_1+q_2=m} a_i^{q_1} a_j^{q_2} - (n \leftrightarrow m) \right) \frac{\partial}{\partial a_i} + \mathcal{O}(q^2). \quad (4.60)$$

Similarly, using (4.54) we find

$$[L_n, J_m] = (n - m)J_{n+m} + \frac{Nq}{\prod_{l \neq i} a_{il}^2} \left(W'''(a_i) a_i^{n+m+2} - 2W''(a_i) a_i^{m+1} \sum_{j \neq i} \frac{a_i^{n+1} - a_j^{n+1}}{a_{ij}} \right) \frac{\partial}{\partial a_i} + \mathcal{O}(q^2) \quad (4.61)$$

and

$$[J_n, J_m] = N^2 q^2 \left[(m - n) \frac{W''(a_i)^2 a_i^{n+m+1}}{\prod_{l \neq i} a_{il}^4} + \frac{2W''(a_i)}{\prod_{l \neq i} a_{il}^2} \left(a_i^{m+1} \sum_{j \neq i} \frac{W''(a_j) a_j^{n+1}}{a_{ij} \prod_{l \neq j} a_{jl}^2} - (n \leftrightarrow m) \right) \right] + \mathcal{O}(q^3). \quad (4.62)$$

An interesting feature of the above equations is to show explicitly that the algebra does not close, as discussed in 4.3: the quantum corrections would have to be linear combinations of the operators at lower order, which is impossible due to the pole structure.

5. Outlook

In this paper, following [1], we have provided a detailed microscopic analysis of the $\mathcal{N} = 1$ gauge theory with one adjoint chiral multiplet and arbitrary tree-level superpotential. We have shown how to use Nekrasov's instanton technology to derive many deep results in $\mathcal{N} = 1$ gauge theories. In particular, we have provided the first non-perturbative discussion of the generalized Konishi anomaly equations, putting forward the subtle constraints coming from working at finite N and deriving the strong quantum corrections to the operators that generate them. We have also computed explicitly the first two terms in the instanton expansion of various operators in the Ω -background, including the generating function $S(z; \mathbf{a}, \mathbf{g}, q)$ for the generalized glueball operators.

Our calculations were limited to the two-instantons order. A full solution of the problem, which includes in particular the calculation of the function S and the derivation of the equations (4.38) and (4.39) is of course highly desirable. It will be presented in a forthcoming publication [12]. The fact that the present microscopic formalism, based on the sum over colored partitions, can match the results from the matrix model approach is a very deep property, clearly related to the open/closed string duality.

It would also be extremely interesting to study the theory with flavors of fundamental quarks and other models with various gauge groups and matter contents along the same line. It seems that the derivation, from a direct microscopic analysis, of all the conjectured exact results in $\mathcal{N} = 1$ gauge theories is now at hand. After almost fifteen years of intense study of the non-perturbative properties of these theories, we believe that this is a highly satisfactory result.

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A. Technical appendix

In this appendix, we prove the equivalence between the formulas (2.32) and (2.33) for the measure on the set of colored partitions. Both formulas have appeared in the literature, starting from [2], but often in erroneous or undeterminate forms (for example by writing them in terms of ambiguous infinite products). Since having the exact formulas was essential to perform our explicit calculations, we have been extremely careful in deriving them and we hope that this appendix will clarify the main properties of the measure factor.

We shall need the following simple

Lemma. *Let k be a partition and $z \in \mathbb{C}$. Then*

$$\frac{1}{z - k_1} \prod_{\beta=1}^{k_1} \frac{z + \tilde{k}_\beta - \beta}{z + \tilde{k}_\beta - \beta + 1} = \frac{1}{z + \tilde{k}_1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - k_\alpha}{z + \alpha - k_\alpha - 1}. \quad (\text{A.1})$$

The proof is made recursively on the number of columns of the partition k . We first consider a partition whose Young tableau Y_k has a single column of arbitrary length, i.e. $k_\alpha = 1$ for $1 \leq \alpha \leq \tilde{k}_1$. In this case, the left hand side of (A.1) reads

$$\frac{1}{z - 1} \frac{z + \tilde{k}_1 - 1}{z + \tilde{k}_1}, \quad (\text{A.2})$$

consistently with the right hand side which, using the many cancellations between the numerator and the denominator in the product, reads

$$\frac{1}{z + \tilde{k}_1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - 1}{z + \alpha - 2} = \frac{1}{z + \tilde{k}_1} \frac{z + \tilde{k}_1 - 1}{z - 1}. \quad (\text{A.3})$$

Now, we assume that the lemma is true for partitions k with k_1 columns in the Young tableau. Let us consider a partition k' with $k'_1 = k_1 + 1$ columns. Its Young tableau $Y_{k'}$ can be built by adding its first column to a Young tableau Y_k having only k_1 columns. Precisely, we have $k'_\alpha = k_\alpha + 1$ for $1 \leq \alpha \leq \tilde{k}_1$ and $k'_\alpha = 1$ for $\tilde{k}_1 + 1 \leq \alpha \leq \tilde{k}'_1$. The left hand side of (A.1) for k' is

$$\begin{aligned} \frac{1}{z - k'_1} \prod_{\beta=1}^{k'_1} \frac{z + \tilde{k}'_\beta - \beta}{z + \tilde{k}'_\beta - \beta + 1} &= \frac{1}{z - k_1 - 1} \prod_{\beta=1}^{k_1+1} \frac{z + \tilde{k}'_\beta - \beta}{z + \tilde{k}'_\beta - \beta + 1} \\ &= \frac{1}{z - k_1 - 1} \frac{z + \tilde{k}'_1 - 1}{z + \tilde{k}'_1} \prod_{\beta=1}^{k_1} \frac{z + \tilde{k}_\beta - \beta - 1}{z + \tilde{k}_\beta - \beta}. \end{aligned} \quad (\text{A.4})$$

In the second line of (A.4) we have explicitly splitted the product over β into the term $\beta = 1$ and the product over $2 \leq \beta \leq k_1 + 1$ for which we can use $\tilde{k}'_\beta = \tilde{k}_{\beta-1}$. Using the recursion hypothesis for k with $z - 1$ replacing z , we can compute the product over β in the second line of (A.4), which yields

$$\frac{1}{z - k'_1} \prod_{\beta=1}^{k'_1} \frac{z + \tilde{k}'_\beta - \beta}{z + \tilde{k}'_\beta - \beta + 1} = \frac{z + \tilde{k}'_1 - 1}{z + \tilde{k}'_1} \frac{1}{z + \tilde{k}_1 - 1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - k_\alpha - 1}{z + \alpha - k_\alpha - 2}. \quad (\text{A.5})$$

On the other hand, we compute the right hand side of (A.1) for k' by splitting the product over α into two terms as

$$\frac{1}{z + \tilde{k}'_1} \prod_{\alpha=1}^{\tilde{k}'_1} \frac{z + \alpha - k'_\alpha}{z + \alpha - k'_\alpha - 1} = \frac{1}{z + \tilde{k}'_1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - k_\alpha - 1}{z + \alpha - k_\alpha - 2} \prod_{\alpha=\tilde{k}_1+1}^{\tilde{k}'_1} \frac{z + \alpha - 1}{z + \alpha - 2}. \quad (\text{A.6})$$

Using the many cancellations in the above products, we find

$$\frac{1}{z + \tilde{k}'_1} \prod_{\alpha=1}^{\tilde{k}'_1} \frac{z + \alpha - k'_\alpha}{z + \alpha - k'_\alpha - 1} = \frac{1}{z + \tilde{k}'_1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - k_\alpha - 1}{z + \alpha - k_\alpha - 2} \frac{z + \tilde{k}'_1 - 1}{z + \tilde{k}_1 - 1}, \quad (\text{A.7})$$

matching with (A.5), which proves the lemma.

A useful corollary of (A.1) is that, for any integer $K \geq 0$,

$$\prod_{\beta=1}^{k_1} \frac{z + \tilde{k}_\beta - \beta}{z + \tilde{k}_\beta - \beta + K} = \prod_{\beta'=1}^K \frac{z + \beta' - 1 - k_1}{z + \beta' - 1 + \tilde{k}_1} \prod_{\alpha=1}^{\tilde{k}_1} \frac{z + \alpha - k_\alpha + K - 1}{z + \alpha - k_\alpha - 1}. \quad (\text{A.8})$$

This identity is very useful to relate products over the columns of a Young tableau to products over the rows of the same tableau, which is exactly what is needed to go from (2.32) to (2.33). Using the notation (3.11), let us rewrite (2.32) and (2.33) as

$$\mu_{\vec{k}} = \prod_{i=1}^N \mu_{k_i} \prod_{i < j} \nu_{\vec{k}, ij} = \prod_{i=1}^N \mu_{k_i} \prod_{i < j} \kappa_{\vec{k}, ij} \quad (\text{A.9})$$

with

$$\nu_{\vec{k}, ij} = \prod_{\square_{(\alpha, \beta)} \in Y_{k_i}} \frac{1}{a_{ij} + \epsilon(\beta - \alpha)} \prod_{\square_{(\alpha, \beta)} \in Y_{k_j}} \frac{-1}{a_{ij} + \epsilon(\alpha - \beta)} \times \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{j,1}} \frac{(a_{ij} + \epsilon(\tilde{k}_{j,\beta} - \alpha - \beta + 1))(a_{ij} + \epsilon(k_{i,\alpha} - \beta - \alpha + 1))}{(a_{ij} + \epsilon(1 - \alpha - \beta))(a_{ij} + \epsilon(\tilde{k}_{j,\beta} - \alpha + k_{i,\alpha} - \beta + 1))}, \quad (\text{A.10})$$

$$\kappa_{\vec{k}, ij} = (-1)^{|k_j|} \prod_{\alpha_1=1}^{\tilde{k}_{i,1}} \prod_{\alpha_2=1}^{\tilde{k}_{j,1}} \frac{a_{ij} + \epsilon(k_{i,\alpha_1} - k_{j,\alpha_2} - \alpha_1 + \alpha_2)}{a_{ij} + \epsilon(\alpha_2 - \alpha_1)} \times \prod_{\square_{(\alpha, \beta)} \in Y_{k_i}} \frac{1}{a_{ij} + \epsilon(\beta - \alpha + \tilde{k}_{j,1})} \prod_{\square_{(\alpha, \beta)} \in Y_{k_j}} \frac{1}{a_{ij} - \epsilon(\beta - \alpha + \tilde{k}_{i,1})}. \quad (\text{A.11})$$

We claim that

$$\nu_{\vec{k}, ij} = \kappa_{\vec{k}, ij}, \quad (\text{A.12})$$

which is a slightly stronger result than the equality between (2.32) and (2.33). To prove this claim, we use (A.8) for the partition k_j , with $K = k_{i,\alpha}$ and $z = a_{ij}/\epsilon - \alpha + 1$. This yields

$$\prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{j,1}} \frac{a_{ij} + \epsilon(\tilde{k}_{j,\beta} - \alpha - \beta + 1)}{a_{ij} + \epsilon(\tilde{k}_{j,\beta} - \alpha + k_{i,\alpha} - \beta + 1)} = \prod_{\square_{(\alpha, \beta)} \in Y_{k_i}} \frac{a_{ij} + \epsilon(\beta - \alpha - k_{j,1})}{a_{ij} + \epsilon(\beta - \alpha + \tilde{k}_{j,1})} \times \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\alpha'=1}^{\tilde{k}_{j,1}} \frac{a_{ij} + \epsilon(k_{i,\alpha} - k_{j,\alpha'} - \alpha + \alpha')}{a_{ij} + \epsilon(-k_{j,\alpha'} - \alpha + \alpha')}. \quad (\text{A.13})$$

Moreover, it is straightforward to check the following identities, that are obtained using the many cancellations between the numerators and the denominators in the right hand side of the equations,

$$\prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{j,1}} \frac{a_{ij} + \epsilon(k_{i,\alpha} - \beta - \alpha + 1)}{a_{ij} + \epsilon(1 - \beta - \alpha)} = \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{j,1}} \prod_{\beta'=1}^{k_{i,\alpha}} \frac{a_{ij} + \epsilon(\beta' - \alpha - \beta + 1)}{a_{ij} + \epsilon(\beta' - \alpha - \beta)} = \prod_{\square_{(\alpha, \beta)} \in Y_{k_i}} \frac{a_{ij} + \epsilon(\beta - \alpha)}{a_{ij} + \epsilon(\beta - \alpha - k_{j,1})}, \quad (\text{A.14})$$

$$\begin{aligned}
 \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\alpha'=1}^{\tilde{k}_{j,1}} \frac{a_{ij} + \epsilon(\alpha' - \alpha)}{a_{ij} + \epsilon(\alpha' - \alpha - k_{j,\alpha'})} &= \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\alpha'=1}^{\tilde{k}_{j,1}} \prod_{\beta=1}^{k_{j,\alpha'}} \frac{a_{ij} + \epsilon(\alpha' - \alpha - \beta + 1)}{a_{ij} + \epsilon(\alpha' - \alpha - \beta)} \\
 &= \prod_{\square_{(\alpha,\beta)} \in Y_{\tilde{k}_j}} \frac{a_{ij} + \epsilon(\alpha - \beta)}{a_{ij} + \epsilon(\alpha - \beta - \tilde{k}_{i,1})}.
 \end{aligned} \tag{A.15}$$

Using (A.13), (A.14) and (A.15) in (A.10), we find (A.12) as we wished.

Let us note that the square of the formula (2.33) can be written elegantly as follows,

$$\begin{aligned}
 \mu_{\vec{k}}^2 &= (-1)^{N|\vec{k}|} \prod_{i,j} \left[\prod_{\alpha_1=1}^{\tilde{k}_{i,1}} \prod_{\alpha_2=1}^{\tilde{k}_{j,1}} \frac{a_{ij} + \epsilon(k_{i,\alpha_1} - k_{j,\alpha_2} - \alpha_1 + \alpha_2)}{a_{ij} + \epsilon(\alpha_2 - \alpha_1)} \right. \\
 &\quad \left. \prod_{\square_{(\alpha,\beta)} \in Y_{\tilde{k}_i}} \frac{1}{a_{ij} + \epsilon(\beta - \alpha + \tilde{k}_{j,1})} \prod_{\square_{(\alpha,\beta)} \in Y_{\tilde{k}_j}} \frac{1}{a_{ij} - \epsilon(\beta - \alpha + \tilde{k}_{i,1})} \right], \tag{A.16}
 \end{aligned}$$

with the rule that the ill-defined terms corresponding to $i = j$ and $\alpha_1 = \alpha_2$ in (A.16) are left out. In this form, the analogy with the $N = 1$ case (2.30), as well as the permutation symmetry (2.34), are obvious.

Let us use the previous results to show that

$$\mu_{\vec{k}}^z(\mathbf{a}, \epsilon) = \mu_{\vec{k}}^z(\mathbf{a}, -\epsilon), \tag{A.17}$$

where \vec{k} is the colored partition dual to \vec{k} . This is shown in two steps. First, from the explicit expression (A.10), it is clear that

$$\nu_{\vec{k},ij}^z(\mathbf{a}, \epsilon) = \nu_{\vec{k},ji}^z(\mathbf{a}, -\epsilon). \tag{A.18}$$

Using (A.12), this is equivalent to

$$\kappa_{\vec{k},ij}^z(\mathbf{a}, \epsilon) = \kappa_{\vec{k},ji}^z(\mathbf{a}, -\epsilon). \tag{A.19}$$

Now, it is immediate to check from (A.11) that

$$\kappa_{\vec{k},ij}^z = \kappa_{\vec{k},ji}^z, \tag{A.20}$$

and thus

$$\kappa_{\vec{k},ij}^z(\mathbf{a}, \epsilon) = \kappa_{\vec{k},ij}^z(\mathbf{a}, -\epsilon). \tag{A.21}$$

Equation (A.17) then immediately follows from (A.9).

This implies that the partition function (2.20) is an even function of ϵ , because the sum of the contributions from a given colored partition and its dual will have this property,

$$Z_{\epsilon}(\mathbf{a}, q, \epsilon) = Z_{\epsilon}(\mathbf{a}, q, -\epsilon). \tag{A.22}$$

Moreover, it can also be shown straightforwardly, doing with sums what we have done with products in (A.14) and (A.15), that equation (2.35) can be rewritten in the form

$$u_{n,\vec{k}} = \sum_{i=1}^N \left[a_i^n + \sum_{\beta=1}^{k_{i,1}} \left((a_i - \epsilon(\tilde{k}_{i,\beta} - \beta + 1))^n - (a_i - \epsilon(\tilde{k}_{i,\beta} - \beta))^n + (a_i + \epsilon\beta)^n - (a_i + \epsilon(\beta - 1))^n \right) \right]. \quad (\text{A.23})$$

This implies that

$$u_{n,\vec{k}}(\mathbf{a}, \epsilon) = u_{n,\vec{k}}(\mathbf{a}, -\epsilon). \quad (\text{A.24})$$

Combining (A.17) and (A.24), we see that correlators built from the scalar operators, which include the glueballs (2.60), are even functions of ϵ . This is non-trivial in the colored partition formalism, but this property must clearly be true in view of the definition (2.12).

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